ON THE REPRESENTATION OF A DISC CAM BY A CHAIN OF ARCS

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ABSTRACT

A special circle algorithm to approximate a cam profile will be discussed. The algorithm is based on two arcs per interval. The locus of connection points appears to be a circle itself. This locus circle is helpful in finding a suitable connection point. As the arcs have the exact position and direction at the interval boundaries, the intermediate behaviour needs approximation criteria for position, direction and curvature. In an elastic follower system the curvature jumps (jerks) are responsible for dynamic position inaccuracy. The required dynamic accuracy determines then the maximum curvature jump and thus the number of intervals. It can be concluded that the arcs should be generated from the view of cam mechanism design within the possibilities of manufacturing.

Keywords: Cam, circle algorithm, manufacturing, dynamics, curvature jump

1. INTRODUCTION

In a cam mechanism the cam is the most characteristic element. The profile of a disc cam, as can be seen in fig.1, bears for instance the information to generate the desired periodic motion $\beta(\alpha)$ of the follower link. The profile of a disc cam is mathematically a closed curve, to be expressed usually in a polar co-ordinate system $R(\varphi)$ attached to the disc plane at the centre of rotation. Basically it is possible to distinguish three profiles:

- The kinematic profile (traced by the centre of the roll),
- The materialized profile (at which the roll contacts),
- The milling tool profile (traced by the centre of the mill during manufacturing).

These three profiles are equidistant curves, which means their distance in normal direction is constant.

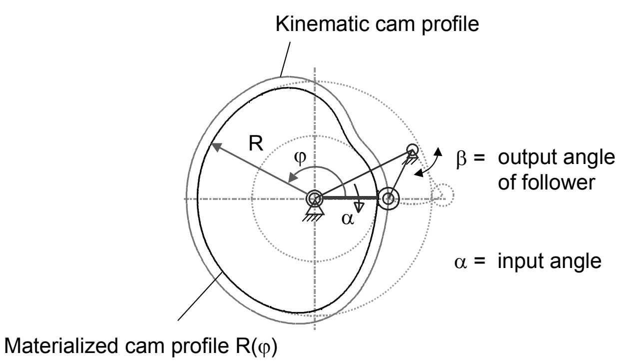

Fig.1 *Cam profiles*

The subject of this paper is the mathematical description of the cam profiles with respect to two applications:

- ♦ Kinematic and dynamic analysis of the follower motion,
- ♦ The manufacturing of the cam.

Both applications require a continuous description of the profiles.

For the manufacturing this is due to the quality of the surface. The milling process provides the best results when the relative speed at the cutting place is constant. To calculate this speed the curve length is needed and this length can actually be obtained in two ways:

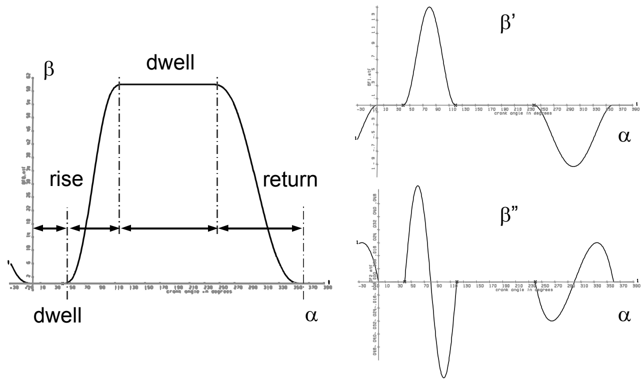

Fig.2 Typical motion definition with continuous functions

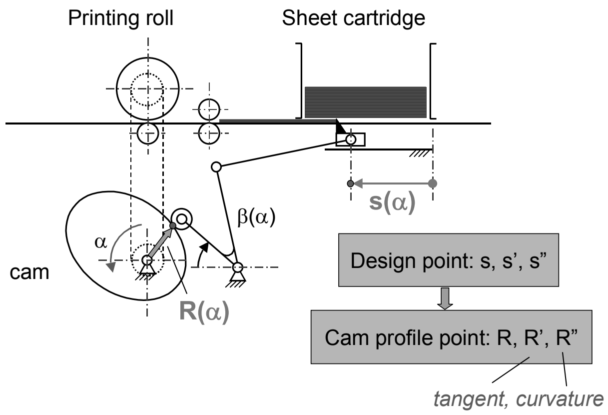

Fig. 3 Cam profile information as available from the design: discrete points and derivatives

- By an exact formula. Here only the straight line and the circle (arc) come into account.
- All other descriptions need numerical integration. Usually such an integration procedure makes use of approximations of the curve by either straight lines or arcs.

For the analysis calculations it looks possible to use the original definition of the design function. Typically such a definition can be done with a chain of standard functions like polynomials. In figure 2 an example is given for a typical dwell-rise-dwell-return motion of the follower link. Such standard functions can be derived at least twice and the connection at the intervals will be continuous up to order two. By backward calculation it is possible then to obtain any point of the profiles, including the tangential direction and the curvature in that point. To obtain the profile in analytical form, it is required then to use the analytical expression of the kinematic transfer function, belonging to the follower mechanism (inverse kinematics). For a simple oscillating follower like in fig. 1 this seems very well possible. In many cases however the follower mechanism is much more complex. Even in the case depicted in fig. 3 is not easy to derive the transfer function $s(\alpha) \rightarrow R(\varphi)$ analytically up to order two. The numerical calculation of the cam profiles, including the derivatives, is anyhow possible, using for instance the Finite Element Method as described in [1]. Besides, the materialized profile is based anyhow on straight lines or arcs.

It can be concluded that a description of the profile by arcs is advantageous, not only during manufacturing, but also to design the profile. The discussion should be focussed then to “how many arcs” in relation to the accuracy criteria to approximate the profile.

2. CIRCLE ALGORITHMS

It will be assumed that the profile description is available in discrete (numerical) form in any amount of points needed for the circle algorithm. In these points also the derivative information (tangent, curvature) is available. The intention is to generate a chain of arcs with which the profile can be approximated. At the connection point of two successive arcs there must be a common tangent (continuity up to order one). Continuity of second order (continuous curvature) is not possible.
Various circle algorithms are known, see fig. 4. They reflect various ideas in profile design.

- Use as few arcs as possible (fig. 4a). The design points will be approximated with a certain tolerance at the profile, to be specified beforehand. The value of the tolerance could be derived from the required position accuracy of the follower link. This algorithm has been described in [2] and will not be discussed further here.
- Use one arc per interval between two design points (fig. 4b). Theoretically this is an option in case the number of profile points is odd. There is precisely one solution for the chain of arcs, which will pass the design points then exactly. The results are sometimes poor because the arcs tend to meander through the points. It is a good idea to consider a tolerance for deviation from the required tangent.

- Two arcs per interval, allowing then generating both the points and the tangent directions exactly (fig. 4c). In this algorithm each interval (pair of arcs) has one degree of freedom. This can be seen as follows: one arc can be chosen at will, and the other circle can be calculated such that it touches the first one. The touching point is the connection point (actually there are two possible connection points). A practical problem of this method is that the choice of the first arc frequently leads to an unexpected connection point. Sometimes the result is improper, for instance when the chain of two arcs has an intersection point.
- Three arcs per interval (fig. 4d). Now it possible to generate also the curvature exactly in both points. This algorithm has also one degree of freedom: the curvature of the connecting arc.

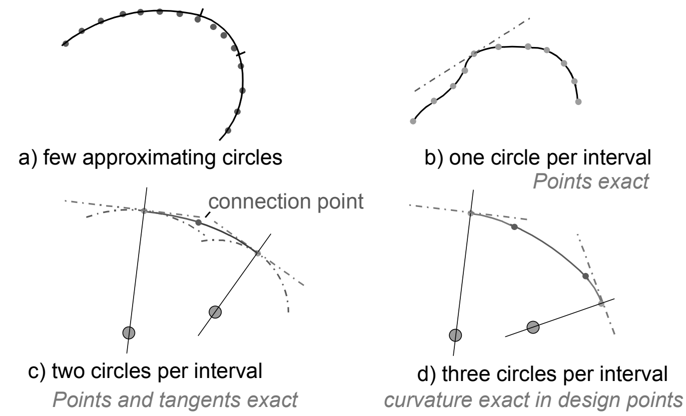

Fig. 4 Circle algorithms

The intention of this paper is to investigate the algorithm of fig. 4c, especially the properties of the connection point. It should be noticed here that, until now, no properties of the connection point have been published in literature.

3. THE ALGORITHM WITH TWO ARCS PER INTERVAL

3.1 First investigation

Struycken [3] had the idea to study a case in which the two points and the two directions were given, see fig. 5. He varied the initial choice of the first radius (in point A) and calculated the second radius (in point B) and the connection points, which he presented in a drawing. The result looks amazing: all connection points seem to lie precisely on a circle through the points A and B. In a more or less intuitive way he succeeded to find a description of this circle, see fig. 6, which he presented as follows:
Intersect the two tangents at A and B to find point C. Then take the bisector of ∠ABC to intersect it with the mid-normal on AB. The result is the centerpoint M of the mentioned circle. Before the proof of this result can be given, a better look at the mathematical description of the arc is required.
Notice that the change of direction from A to B is usually within 180°. The shortest arc provides then the possible solutions for the connection point.

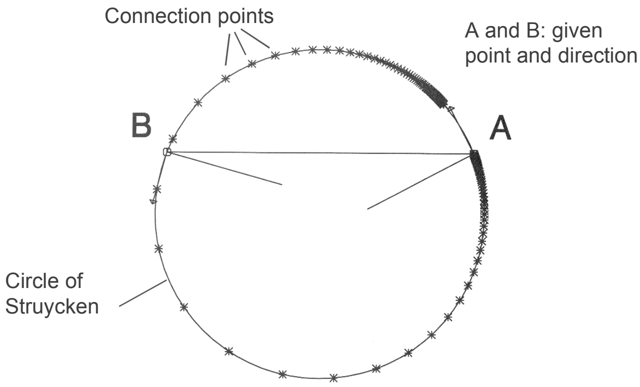

Fig.5 Connection points between the two arcs when the curvature of one arc will be varied

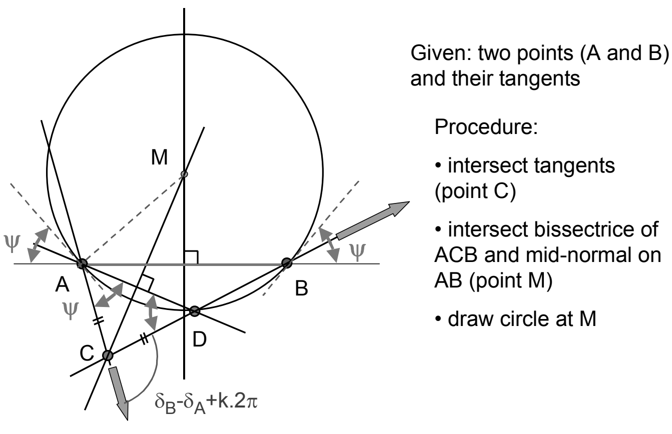

Fig. 6 Geometric construction of the circle, which is the locus of connection points between two arcs (acc. Struycken)

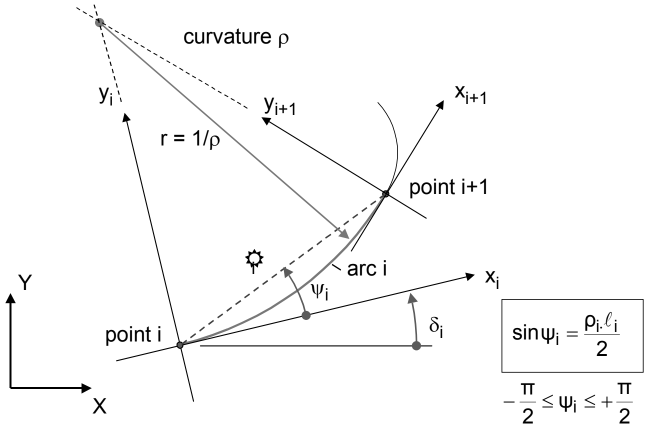

Fig. 7 Special definition of the arc in local co-ordinate system

3.2 Representation of the arc

Frequently the arcs appear to be almost straight lines. The usual description with a radius and a centerpoint is then uncomfortable because of the big numbers necessary to describe it. It looks a better idea to express the arc with the curvature ρ (the reciprocal value of the radius r). Defining the tangent as the abscissa of a local co-ordinate system (x_i, y_i) of arc number i with radius r_i, the equation of the circle would be:

$$(y_i - r_i)^2 + x_i^2 = r_i^2$$

Substituting the curvature ρ = 1/r for the radius and adopting a polar representation by substituting

$$y_i = \ell_i \cdot \sin\psi_i$$

$$x_i = \ell_i \cdot \cos\psi_i$$

the following equation for the arc can be found:

$$\sin\psi_i = \frac{\rho_i \cdot \ell_i}{2} \qquad (1)$$

Note that P_i is the distance between the two points and that angle ψ_i can be applied to calculate the tangent (angle δ) of any connection point of the following arc (i+1)

$$\delta_{i+1} = \delta_i + 2\psi_i \qquad (2)$$

The description includes the straight line (ρ = 0), but is limited with respect to the range of angle ψ_i :

$$-\frac{\pi}{2} \le \psi_i \le +\frac{\pi}{2}$$

The arc can be at its maximum a half circle. The sign of the curvature determines at which side of the chord P_i the arc lies.

3.3 The proof of the locus circle

To prove that the circle of Struycken is indeed the locus of connection points this circle will be represented as an arc according paragraph 3.2. The angle ψ can be detected easily from eq. (2)

$$\psi = \frac{\delta_B - \delta_A}{2} + k.\pi$$

and this angle has been applied in figure 6. The construction of the isosceles triangle ACD provides for instance an alternative graphic way to find the angle ψ.

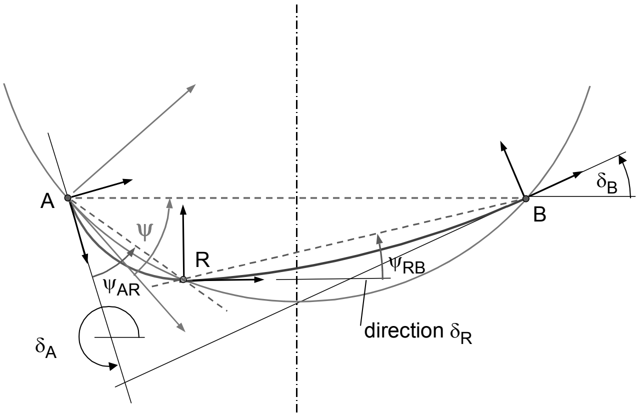

Fig. 8 Connection point R of the two arcs on the circle of Struycken

Consider now an arbitrary point R as the connection point between two arcs between the points A and B, having the directions δ_A and δ_B in A and B respectively, see fig. 8. It should be investigated then when the tangential directions at both arcs are the same in point R. Observing that these two directions can be expressed with eq. (2) as

$$\delta_R = \delta_A + 2\psi_{AR} \quad \text{and} \quad \delta_R = \delta_B - 2\psi_{RB} \text{ respectively,}$$

the difference becomes

$$\delta_A - \delta_B + 2\psi_{AR} + 2\psi_{RB}$$

This difference will become zero (or π) when

$$\psi_{AR} + \psi_{RB} = \frac{\delta_B - \delta_A}{2} = \psi$$

This condition can easily be transferred to $\angle ARB$ = constant, and this is a property of a circle. Point R must lie thus on a circle

4. APPLICATION OF THE ALGORITHM

The locus circle of Struycken provides a help to make a choice for the connection point on each interval. For a successful application two aspects need attention:

- ♦ The best (optimal) choice of the connection point. Some criterion is needed to decide. In this paper no further attention is given to this aspect. It will be assumed that the point on the mid-normal of each chord is usually a fair choice.
- ♦ The number of design points (intervals) needed to describe the profile with sufficient accuracy. Now it is necessary to look better at the criteria that should be applied.

It will be assumed that design points (and derivative information) are available in any amount. So it will be possible to scan a certain interval in any number of points. If the proposed pair of arcs fulfils all criteria, then this interval can be accepted. Otherwise it can for instance be split up in two or more intervals (intermediate design points must be incorporated then). In this paper the criteria will be focussed on.

Criteria can be formulated on

- Position accuracy. Actually the error in radial direction can be calculated. The error is also dependent on the (first order) transfer function of the follower mechanism. In addition to figure 3 the error of the output link Δs can be expressed as

$$\Delta s \mathrel{\hat{=}} \frac{ds}{d\beta} \cdot \frac{d\beta}{dR} \cdot \Delta R \tag{3}$$

  It should be realized that a criterion depends also on the follower mechanism.

- Tangent accuracy. The tangential direction of a curve defined in polar co-ordinates $R(\varphi)$ can be expressed as:

$$\delta = \varphi + \arctan \frac{R'(\varphi)}{R(\varphi)} \qquad (4)$$

In practice however a useful interpretation of such a criterion is absent.

- Curvature accuracy. In an algorithm with arcs the jump in the curvature looks very interesting, especially for the dynamic response. In each connection point between two arcs a jerk will occur perpendicular to the profile. The effect can be estimated with the help of figure 9 (left). A point moving with velocity v along an arc with curvature ρ has a normal acceleration component ρv^2, so a jump in the curvature causes a jump in the normal acceleration component. The same acceleration jump exists when the profile rotates and the point moves for simplicity just in radial direction (fig. 9 right). The jerk at the roll can be specified now

$$\Delta\ddot{h} = (\rho_1 - \rho_2) \cdot R\dot{\alpha}^2$$

For the response of the output link, like the slider in fig. 3, an elastic follower system should be considered. For most applications such an elastic system can be reduced to a simple system with one mass and one spring, see fig. 9. The amplitude Δs caused by the jerk depends on the natural frequency ω_0 of the mass-spring system:

$$\Delta s = \frac{\Delta\ddot{h}}{\omega_0^2}$$

The dynamic accuracy can thus be estimated directly with the curvature jump and the (lowest) natural frequency of the follower. The allowable curvature jump is then

$$\Delta s = \frac{\dot{\alpha}^2}{\omega_0^2} R \cdot \Delta\rho \qquad (5)$$

A discussion on the influence of the (lowest) natural frequency makes sense. In designing step-dwell motions as drawn in fig. 2 usually the oscillation time will be chosen typically in a certain rate to the step time. A factor 7 or more will be recommended for a proper dynamic behaviour [4]. It seems very well possible that a division of the step interval into a comparable number of arcs may occur. In that case it could be possible to generate the jerks in the natural frequency of the follower system, and the dynamic response would be very poor. It can be remarked however that the two arcs at the interval will not have the same length when they have different curvature. The choice of the connection point on the mid-normal of the chord is therefore not bad: the frequency of the jerks tend to be irregular and will usually not give problems with respect to the natural frequency. In special cases the connection points could be taken more randomly. The free choice of the connection point on the locus circle of Struycken offers this opportunity.

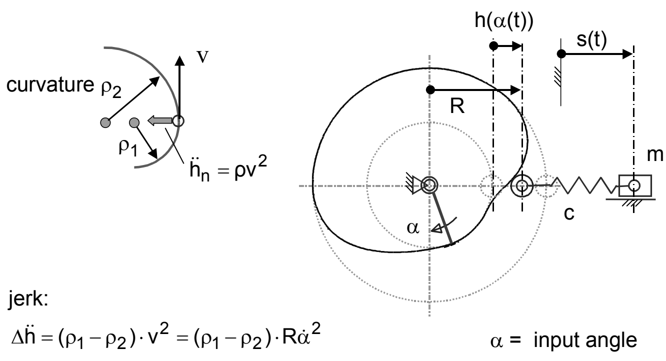

Fig.9 Dynamic response of elastic follower system

5. CONCLUSIONS

The circle algorithm with two arcs per interval has a locus of connection points, which is a circle itself. This property is a help to choose a connection point.

The two arcs, which approximate the profile, must be considered with respect to the design demands of the follower motion, both static and dynamic. It is very well possible to estimate the influence of the profile approximation, including the jerk in the connection points, on the follower motion. Indirectly the dynamic motion requirements determine the required amount of arcs to approximate the profile. It is therefore better to calculate the arcs during the design of the cam and pass the information to the manufacturing process.

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